

# Critical gravity waves

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**ABSTRACT:** Critical Gravity in  $D$  dimensions is discussed from the point of view of its exact solutions. The special features that certain type of solutions of higher-curvature gravity develop when one approaches the critical point of the parameter space are reviewed. In particular, a non-linear realization of the logarithmic modes of linearized Critical Gravity is seen to emerge as a peculiarity of the sector of anti-de Sitter wave solutions. Logarithmic solutions are shown to occur at a second point of the parameter space, at which the effective mass of the anti-de Sitter waves equals the Breitenlohner-Freedman bound. Other type of solutions with anisotropic scale invariance are also discussed and the special features they develop at the critical point are studied as well.

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**KEYWORDS:** Higher-curvature gravity, Higher-dimensional gravity.

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## 1. Critical gravity

In reference [1], Lü and Pope addressed the problem of whether a four-dimensional analogue of the three-dimensional chiral gravity constructed in [2] exists<sup>1</sup>. The strategy was to consider square-curvature modifications to four-dimensional Einstein-Hilbert action with negative cosmological constant, and demand the coefficients of the action to take specific values to exclude the ghost-like excitations. While a fine tuning between the two different coefficients of the square-curvature action is enough to decouple the scalar excitations, demanding the cosmological constant to take a specific value leads the massive graviton-like excitations to become massless. As a consequence, the square-curvature theory with negative cosmological constant renders free of ghosts about its anti-de Sitter (AdS) background, although, as it happens in three dimensions, logarithmic modes that may spoil unitarity appear.

The construction of [1] has received the name of Critical Gravity, and it was subsequently extended to  $D > 4$  dimensions in reference [7]; other extensions, like its supersymmetric version and its higher-curvature analogues, were discussed in [8, 9, 10] and references thereof. Below we will discuss Critical Gravity in dimension  $D = 4$  and higher in detail and study special sector of its space of solutions.

An interesting aspect of Critical Gravity, and which has shown to be crucial to investigate the consistency of the theory as a toy model of quantum gravity [11], is the existence of spin-2

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<sup>1</sup>See [3, 4, 5, 6] for discussions about the three-dimensional model.

logarithmic modes [1]. These modes, hereafter referred to as log-modes, are analogous to those found in Topologically Massive Gravity at the chiral point<sup>2</sup> [5, 6], and within the context of the four-dimensional model these were also studied in [13, 14, 15]. Here, we will analyze the log-modes of  $D$ -dimensional Critical Gravity at the level of the exact solutions. Our analysis is based on our previous works [16, 17] and is close to the analysis done in reference [15].

Let us begin by reviewing Critical Gravity in  $D \geq 4$  dimensions.

### 1.1 Critical Gravity in $D = 4$ dimensions

Critical Gravity in  $D = 4$  is defined by the action [1]

$$S[g_{\mu\nu}] = \int d^4x \sqrt{-g} \left( R - 2\Lambda + \beta_1 R^2 + \beta_2 R_{\alpha\beta} R^{\alpha\beta} \right). \quad (1.1)$$

with its coupling constants restricted to obey the relation

$$\beta_2 = -3\beta_1 \quad (1.2)$$

together with

$$\beta_1 = -\frac{1}{2\Lambda}. \quad (1.3)$$

The relation (1.2) between the coupling constants  $\beta_i$  implies that the coefficient of the  $\square R$  contribution to the trace of the equations of motion vanishes; see (1.7) below. Then, at this point of the parameter space the theory loses its scalar mode. On the other hand, while condition (1.2) is needed to decouple the spin-0 excitation, condition (1.3) is needed for the spin-2 excitations of the theory to become massless. The latter condition may, however, be relaxed and, following [9], the model with the cosmological constant taking values in the range  $0 < \beta_1 \leq -1/(2\Lambda)$  may be considered. The model of [1] then appears as the upper bound of this window.

Because of the Gauss-Bonnet theorem, we know that in four dimensions the Kretschmann scalar  $R_{\alpha\beta\gamma\delta} R^{\alpha\beta\gamma\delta}$  can be written in terms of the square-curvature terms appearing in (1.1), up to a total derivative. That is, action (1.1) is the most general quadratic action in three and four dimensions. As said, Critical Gravity is defined by considering (1.1) with (1.2)-(1.3), and in many ways it is analogous to the three-dimensional chiral gravity introduced in [2] and its generalization that includes the model proposed in [12]. The main goal of the present work is to emphasize this analogy by establishing a parallelism between our previous work [16] and the  $D \geq 4$  critical models.

### 1.2 Extension to $D > 4$ dimensions

Unlike the three- and the four-dimensional cases, in dimension  $D > 4$  three different invariants have to be used to write the most general quadratic action; namely

$$S[g_{\mu\nu}] = \int d^Dx \sqrt{-g} \left( R - 2\Lambda + \beta_1 R^2 + \beta_2 R_{\alpha\beta} R^{\alpha\beta} + \beta_3 R_{\alpha\beta\mu\nu} R^{\alpha\beta\mu\nu} \right). \quad (1.4)$$

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<sup>2</sup>and in its analogue, the New Massive Gravity of [12].

Critical Gravity in  $D$ -dimensions corresponds to considering this action with the following tuning of the coupling constants

$$\beta_1 = \frac{2\beta_3}{(D-1)(D-2)}, \quad \beta_2 = -\frac{4\beta_3}{(D-2)}, \quad (1.5)$$

and also

$$\Lambda = \frac{(D-1)(D-2)}{8\beta_3(D-3)}. \quad (1.6)$$

The linear combination (1.5) of the quadratic terms in (1.4) is such that, up to a total derivative, it coincides with  $\sqrt{-g}C_{\alpha\beta\gamma\delta}C^{\alpha\beta\gamma\delta}$ , where  $C_{\alpha\beta\gamma\delta}$  is the Weyl tensor; in four dimensions, this modification is conformally invariant.

The strategy here will be first considering the general quadratic gravity action and then see what special features appear at the critical point. In turn, let us begin by writing the field equations that correspond to vary (1.4) with respect to the metric,

$$\begin{aligned} R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} + \Lambda g_{\mu\nu} - \frac{1}{2} \left( \beta_1 R^2 + \beta_2 R_{\alpha\beta} R^{\alpha\beta} + \beta_3 R_{\alpha\beta\gamma\delta} R^{\alpha\beta\gamma\delta} \right) g_{\mu\nu} \\ + 2\beta_3 R_{\mu\gamma\alpha\beta} R_{\nu}{}^{\gamma\alpha\beta} + 2(\beta_2 + 2\beta_3) R_{\mu\alpha\nu\beta} R^{\alpha\beta} - 4\beta_3 R_{\mu\alpha} R_{\nu}{}^{\alpha} + 2\beta_1 R R_{\mu\nu} \\ + (\beta_2 + 4\beta_3) \square R_{\mu\nu} + \frac{1}{2} (4\beta_1 + \beta_2) g_{\mu\nu} \square R - (2\beta_1 + \beta_2 + 2\beta_3) \nabla_{\mu} \nabla_{\nu} R = 0. \end{aligned} \quad (1.7)$$

We already notice from this that mode  $\square R$  actually decouples as it does not appear in the trace of the field equations.

Before deriving different classes of solutions to (1.7), let us first fix the cosmological constant  $\Lambda$  such that the  $\text{AdS}_D$  spacetime of radius  $l$  is a solution of the equations (1.7). Then, we find the following constraint between the cosmological constant  $\Lambda$ , the  $\text{AdS}_D$  radius  $l$ , and the coupling constants  $\beta_i$

$$\Lambda = -\frac{(D-1)(D-2)}{2l^2} + \frac{(D-1)(D-4)}{2l^4} [(D-1)(D\beta_1 + \beta_2) + 2\beta_3]. \quad (1.8)$$

From this, we notice that only in four dimensions the cosmological constant is related to the AdS radius in the usual way, without involving the couplings  $\beta_i$ . For  $D \neq 4$  the value of  $l$  is uniquely given in terms of  $\Lambda$  for very special values of  $\beta_i$ ; see below. In the generic case, there are two maximally symmetric backgrounds provided the effective cosmological constant  $-l^{-2}$  may take two different values. In Critical Gravity (1.5)-(1.6), on the contrary, the second term on the right hand side of (1.8) vanishes and thus the vacuum is unique.

## 2. Anti-de Sitter waves

Here, we will explore exact solutions to Critical Gravity in  $D$  dimensions. We consider an ansatz of the following form

$$ds^2 = \frac{l^2}{r^2} \left[ -(1+2h) dt^2 + 2dt d\xi + dr^2 + d\vec{x}^2 \right], \quad (2.1)$$

where  $h$  is a metric function that does not depend on the lightlike coordinate  $\xi$ . Here,  $d\vec{x}^2$  refers to the Euclidean metric on the  $\mathbb{R}^{D-3}$  plane. We will consider deformations of the universal covering of  $\text{AdS}_D$ , so the timelike coordinate takes values  $t \in \mathbb{R}$ , as well as the lightlike coordinate  $\xi \in \mathbb{R}$ . The radial coordinate takes values  $r \in \mathbb{R}_{\geq 0}$ . For  $h = \text{const}$ , we recover the metric of  $\text{AdS}_D$  space in Poincaré coordinates, where the boundary of  $\text{AdS}_D$  is located at  $r = 0$ . Metric (2.1) corresponds to a special class of the so-called Siklos spacetimes [18]; and in the general case, solutions (2.1) can be thought of as describing exact gravitational waves propagating on the  $\text{AdS}_D$  spacetime [19] (hereafter referred to as AdS-waves). Solutions (2.1) are, indeed, a special case of a more general family of solutions studied in [20].

Here, we will be mostly concerned with AdS-wave solutions of the type (2.1) in the case that  $h$  only depends on the radial coordinate  $r$ . Nevertheless, it is worth mentioning that most of the formulae below remains valid when the profile of the wave corresponds to a more general Siklos solution with  $h = h(t, r, \vec{x})$ . We use the null geodesic vector  $k^\mu \partial_\mu = (r/l) \partial_\xi$  that allows us to interpret these backgrounds as generalized Kerr-Schild transformations of  $\text{AdS}_D$ ; namely

$$g_{\mu\nu} = g_{\mu\nu}^{\text{AdS}} - 2h k_\mu k_\nu. \quad (2.2)$$

where  $g_{\mu\nu}^{\text{AdS}}$  is the metric of  $\text{AdS}_D$ , and recall  $k_\mu k^\mu = 0$ .

The Ricci tensor for a metric like (2.2) takes the form

$$R_{\mu\nu} = -\frac{(D-1)}{l^2} g_{\mu\nu} + k_\mu k_\nu \square h, \quad (2.3)$$

and it yields constant scalar curvature  $R = -D(D-1)/l^2$ , exactly the same as for  $\text{AdS}_D$  space. It also gives the squared-curvature combinations

$$R_{\mu\alpha} R_\nu{}^\alpha = \frac{(D-1)^2}{l^4} g_{\mu\nu} - \frac{2(D-1)}{l^2} k_\mu k_\nu \square h, \quad (2.4)$$

$$R_{\mu\alpha\nu\beta} R^{\alpha\beta} = \frac{(D-1)^2}{l^4} g_{\mu\nu} - \frac{(D-2)}{l^2} k_\mu k_\nu \square h, \quad (2.5)$$

$$R_{\mu\gamma\alpha\beta} R_\nu{}^{\gamma\alpha\beta} = \frac{2(D-1)}{l^4} g_{\mu\nu} - \frac{4}{l^2} k_\mu k_\nu \square h. \quad (2.6)$$

Using the expression for the Ricci tensor (2.3), together with the null and geodesic properties of  $k^\mu$ , one finds that all the invariants constructed from the Riemann tensor turn out to coincide with those of  $\text{AdS}_D$  spacetime. Besides, one finds that the only contribution to the equations of motion that contains derivatives of the Riemann tensor is

$$\square R_{\mu\nu} = k_\mu k_\nu \square \left( \square - \frac{2}{l^2} \right) h. \quad (2.7)$$

Then, taking into account that  $\Lambda$  is given by (1.8), one finally finds that the equations of motion (1.7) reads

$$(\beta_2 + 4\beta_3) k_\mu k_\nu (\square - m^2) \square h = 0. \quad (2.8)$$

where  $m^2$  is given by

$$m^2 = \frac{2(D-1)(D\beta_1 + \beta_2) - 4(D-4)\beta_3 - l^2}{l^2(\beta_2 + 4\beta_3)}. \quad (2.9)$$

We will be specially interested in solutions that tend to  $\text{AdS}_D$  spacetime close to the boundary. In turn, as always when dealing with asymptotically  $\text{AdS}_D$  spaces, a crucial question is that about the next-to-leading dependence in large distances (i.e. large  $1/r$ ) expansion. Assuming that the function  $h$  only depends on the radial coordinate  $r$ , the equations of motion reduces to the single homogeneous Euler differential equation

$$\begin{aligned} & (\beta_2 + 4\beta_3) [r^2 h'''' - 2(D-4)rh'''] + \\ & + [l^2 - 2D(D-1)\beta_1 + (D-2)(D-8)\beta_2 + 4(D-2)(D-5)\beta_3] h'' - \\ & - (D-2) [l^2 - 2D(D-1)\beta_1 - (3D-4)\beta_2 - 8\beta_3] \frac{1}{r} h' = 0, \end{aligned} \quad (2.10)$$

where the prime stands for the derivative with respect to  $r$ ,  $f' = \partial_r f$ , and where we used that  $\square f = [r^2 f'' - (D-2)rf'] / l^2$ .

Equation (2.10) actually corresponds to the  $tt$  component of the equations of motion, which is the only non-trivial one. In what follows, we provide a detailed analysis of the equation (2.10) and its solutions; in particular, we will see that non-trivial asymptotically  $\text{AdS}_D$  solutions exist.

## 2.1 Massless waves of the second order sector

Let us begin by noticing that, although in the generic case (2.10) is a fourth order linear differential equation, in the particular case  $\beta_2 = -4\beta_3$  the first line in (2.10) disappears and then the equation renders of second order, yielding

$$[l^2 - 2D(D-1)\beta_1 + 12(D-2)\beta_3] \square h = 0; \quad (2.11)$$

notice that  $\square$ , which corresponds to the d'Alembertian operator associated to metric (2.1), does not depend on the function  $h$ , so that in particular it coincides with the d'Alembertian operator in  $\text{AdS}_D$  spacetime.

If in addition to  $\beta_2 = -4\beta_3$  we considered  $\beta_1 = [l^2 + 12(D-2)\beta_3] / [2D(D-1)]$  then full degeneracy would occur and the field equations would then be satisfied for any profile function  $h$ . This degenerate point of the space of parameter is also interesting and, for instance, in  $D = 5$  dimensions it includes the five-dimensional Chern-Simons gravity theory for the group  $\text{SO}(2, 4)$ , for which this type of degeneracy is known to exist. On the contrary, for  $\beta_1 \neq [l^2 + 12(D-2)\beta_3] / [2D(D-1)]$ , the resulting equation is strictly of second order, as in the case of Lovelock theory [21]. In fact, Lovelock theory is a particular case of the former, which corresponds to having  $\beta_2 = -4\beta_3$  together with  $\beta_1 = \beta_3$ .

In general, in the second order case  $\beta_2 = -4\beta_3$  the solution is given by

$$h(r) = c_0 r^{D-1}, \quad (2.12)$$

up to an arbitrary constant that can always be removed by a change of coordinates. Not surprisingly, (2.12) coincides with a solution of General Relativity (GR) with negative cosmological constant, i.e. the theory that corresponds to  $\beta_i = 0$ . This solution is, indeed, the usual GR exact massless scalar mode that propagates on  $\text{AdS}_D$  spacetime, and the field equation (2.11) in that case reduces to the wave equation  $\square h = 0$ , with  $\square$  being the d'Alembertian operator of  $\text{AdS}_D$ .

Generically, a solution like (2.1) is physically interpreted as an exact massive gravitational wave propagating on the  $\text{AdS}_D$  spacetime also in the cases  $\beta_i \neq 0$ . In fact, for generic values of  $\beta_2 \neq -4\beta_3$ , one finds that at least one the set of solutions to (2.8) obeys the Klein-Gordon wave function  $(\square - m^2)h = 0$  with an effective mass parameter  $m$  being given by (2.9). We will explore these solutions in detail below.

## 2.2 Massive waves in anti-de Sitter spacetime

For  $\beta_2 \neq -4\beta_3$ , field equation (2.10) is a fourth-order Euler differential equation which, in the generic case, admits linearly independent solutions of the power-law form  $h \sim r^\alpha$ , with the exponents  $\alpha$  being the roots of the fourth-degree polynomial

$$\alpha(\alpha - D + 1) \left[ \left( \alpha - \frac{D-1}{2} \right)^2 - \frac{(D-1)^2}{4} - \frac{l^2 - 2(D-1)(D\beta_1 + \beta_2) + 4(D-4)\beta_3}{\beta_2 + 4\beta_3} \right] = 0. \quad (2.13)$$

Since the constant solution, i.e.  $\alpha = 0$ , can be removed by coordinate transformations, the general solution results to be given by

$$h(r) = c_0 r^{D-1} + c_+ r^{\alpha_+} + c_- r^{\alpha_-}, \quad (2.14)$$

where

$$\alpha_{\pm} = \frac{D-1}{2} \pm \sqrt{\frac{(D-1)^2}{4} + \frac{2(D-1)(D\beta_1 + \beta_2) - 4(D-4)\beta_3 - l^2}{\beta_2 + 4\beta_3}}, \quad (2.15)$$

and where  $c_0$  and  $c_{\pm}$  are arbitrary integration constants.

The type of solutions (2.14) includes cases which are interesting for physics. For instance, for  $c_0 = c_+ = 0$  it includes geometries whose isometry group is the so-called Schrödinger group, namely solutions of the form

$$ds^2 = \frac{l^2}{r^2} \left( -\frac{1}{r^{2z-2}} dt^2 + 2dt d\xi + dr^2 + d\vec{x}^2 \right), \quad (2.16)$$

where  $\alpha_- = 2(1-z)$ . These geometries have recently been considered as holographic gravity duals of non-relativistic systems [22, 23] with Galilean invariance and anisotropic scaling symmetry under  $(t, \xi, r, x) \rightarrow (\lambda^{2z}t, \lambda^{4-2z}\xi, \lambda^2r, \lambda^2x)$ .

Among solutions (2.14) there are also solutions that asymptote  $\text{AdS}_D$  spacetime at large distance. To see this, let us define the new timelike coordinate  $\hat{t} \equiv t - \xi$  and the radial

coordinate  $\hat{r} \equiv l^2/r$ , so that the near boundary region of  $\text{AdS}_D$  corresponds to large  $\hat{r}$ . In these coordinates, metric (2.1) takes the form

$$ds^2 = \frac{\hat{r}^2}{l^2}(-d\hat{t}^2 + d\xi^2 + d\vec{x}^2) + \frac{l^2}{\hat{r}^2}dr^2 - \hat{h}(\hat{r})(d\hat{t} + d\xi)^2, \quad (2.17)$$

where the function  $\hat{h}(\hat{r}) = 2\hat{r}^2 h(1/\hat{r})/l^2$  represents the perturbation of  $\text{AdS}_D$  spacetime. Asymptotically  $\text{AdS}_D$  solutions then correspond to solutions with  $\alpha_+ \geq 2$ , for which

$$g_{\hat{t}\hat{t}} = -\frac{\hat{r}^2}{l^2} + \mathcal{O}(1), \quad g_{\hat{t}\xi} = -\frac{\hat{r}^2}{l^2} + \mathcal{O}(1), \quad g_{\xi\xi} = -\frac{\hat{r}^2}{l^2} + \mathcal{O}(1),$$

where  $\mathcal{O}(1)$  stands for terms that either do not depend on  $\hat{r}$  or whose  $\hat{r}$ -dependences decay near the boundary. Relaxed asymptotic conditions involving logarithmic dependences in the next-to-leading order in the  $1/\hat{r}$  expansion will be discussed below.

As mentioned, solutions (2.14) generically satisfy the wave equation<sup>3</sup>  $(\square - m^2)h = 0$  with  $m$  given by (2.9). More precisely, solutions (2.14) describe the superposition of three scalar modes: the massless mode of GR and two massive modes generated by the square-curvature terms, the latter having the same effective mass (2.9). It is worth pointing out that the solution (2.14) is valid only when the roots (2.15) are real. This in turn constrains the mass parameter to obey strictly the Breitenlohner-Freedman (BF) bound of the  $\text{AdS}_D$  in which the mode propagates [24]; namely

$$m^2 > m_{\text{BF}}^2 \equiv -\frac{(D-1)^2}{4l^2}. \quad (2.18)$$

### 3. Logarithmic Gravity in $D$ dimensions

We now turn to analyze the cases for which the roots  $\alpha_{\pm}$  of the polynomial (2.13) degenerate and two would-be-independent solutions to the differential equation coalesce. As usual, when this type of multiplicity in the roots takes place, the power-law dependence does not represent the more general solutions to Eq. (2.10), and additional behaviors, typically involving logarithmic dependences, appear. The existence of such *exact* logarithmic behaviors is totally analogous to what happens in three-dimensional massive gravity; see for instance [25, 26, 27]; see also [16].

#### 3.1 The Logarithmic sectors I: The Breitenlohner-Freedman point.

The first source of multiplicity appears when the roots (2.15) become one single root, namely when  $\alpha_+ = \alpha_-$ . This happens when

$$\beta_2 = \frac{4[l^2 - 2D(D-1)\beta_1 - (D^2 - 6D + 17)\beta_3]}{(D+7)(D-1)}, \quad (3.1)$$

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<sup>3</sup>Strictly speaking, this is true for generic  $\alpha_{\pm}$ , with  $\alpha_+ \neq \alpha_-$  and  $\alpha_{\pm} \neq D-1$ ; see below.



In this resonant case the mass  $m^2$  tends to the BF bound  $m_{\text{BF}}^2$  and the general solution turns to be

$$h(r) = c_0 r^{D-1} + c_1 r^{\frac{D-1}{2}} \log(r) + c_2 r^{\frac{D-1}{2}}, \quad (3.2)$$

up to an additive constant. It is worth noticing that, while in the case of solutions like (3.2) with  $c_2 \neq 0$ ,  $c_1 = 0$  the profile function  $h$  obeys the Klein-Gordon equation

$$(\square - m_{\text{BF}}^2)h = 0 \quad \text{with} \quad m_{\text{BF}}^2 \equiv -\frac{(D-1)^2}{4l^2}, \quad (3.3)$$

in the case of solutions with  $c_1 \neq 0$  we find

$$(\square - m_{\text{BF}}^2)\square h = 0, \quad (3.4)$$

though  $h$  does not satisfy the Klein-Gordon equation for any value of  $m$ . This is similar to what was observed in three-dimensional massive gravity [16].

Now, let us turn to the case we are interested in, namely the case of Critical Gravity.

### 3.2 The Logarithmic sectors II: The Critical Gravity point.

The second case in which multiplicity of the roots  $\alpha_{\pm}$  occurs is when one of the two generic roots (2.15) either vanishes or takes the value  $D-1$ . Actually, it is easy to check that these two possibilities occur simultaneously, namely  $\alpha_- = 0$  precisely when  $\alpha_+ = \alpha_+^{(\text{GR})} = D-1$ . In this case, the coupling constants obey

$$\beta_2 = \frac{l^2 - 2D(D-1)\beta_1 + 4(D-4)\beta_3}{2(D-1)}, \quad (3.5)$$

and this is actually what happens at the critical point (1.5)-(1.6). The solution with simultaneous double multiplicity is then of the form

$$h(r) = c_0 r^{D-1} + c_1 r^{D-1} \log(r) + c_2 \log(r). \quad (3.6)$$

That is, Critical Gravity in  $D$  dimensions admits logarithmic modes (3.6) as exact solutions, similarly as it happens in three-dimensional massive gravity [16]. Besides, as it happens with the BF logarithmic modes discussed in the previous subsection, only solutions (3.6) having  $c_1 = c_2 = 0$  obey the massless wave equation  $\square h = 0$ . The logarithmic modes, on the contrary, obey the fourth-order equation

$$\square^2 h = 0 \quad \text{although} \quad \square h \neq 0. \quad (3.7)$$

Again, this is totally analogous to what happens with the linearized log-modes of [1]. In fact, we interpret logarithmic solutions (3.6) as the non-linear realization of the former. To see this it is convenient to consider the basis

$$h_{\pm}^{GR} = \frac{(D-1)}{l^2} (r^{D-1} \mp 1), \quad h_{\pm}^{Log} = (r^{D-1} \pm 1) \log(r), \quad (3.8)$$

that is  $h_{\pm}^{Log} \propto h_{\pm}^{GR} \log(r)$ , in which the d’Alambertian operator takes the following Jordan’s cell form

$$\square h_{\pm}^{Log}(r) = h_{\pm}^{GR}(r), \quad \square h_{\pm}^{GR}(r) = 0. \quad (3.9)$$

The question remains as to whether Log-modes can be consistently decoupled by an appropriate choice of boundary conditions. As we learned from the three-dimensional case [6], the question about the consistency of a truncation of the theory may be a subtle one.

#### 4. Other sectors of the space of solutions

The AdS-waves is not the only sector of exact solutions of higher-curvature gravity that exhibits peculiarities at the critical point. In fact, the special character of the critical point can also be observed by investigating other sectors of the space of solutions. For instance, let us consider the so-called Lifshitz metrics in  $D = 4$  dimensions: Consider

$$ds^2 = -\frac{l^{2z}}{r^{2z}} dt^2 + \frac{l^2}{r^2} (dr^2 + d\xi^2 + dx^2), \quad (4.1)$$

where now the coordinate  $\xi$  plays the same role as  $x$ , c.f. (2.16). Metric (4.1) presents anisotropic scale invariance, as the scaling transformation  $(\hat{t}, r, \xi, x) \rightarrow (\lambda^{2z} \hat{t}, \lambda^2 r, \lambda^2 \xi, \lambda^2 x)$  represents a symmetry of it; this is why these spaces have recently been considered in the context of non-relativistic holography as well [28]. Metric (4.1) reduces to  $\text{AdS}_D$  spacetime when  $z = 1$ ; that is, it reduces to the case  $h = \text{const}$  of (2.1).

It was shown in [29] that metric (4.1) is also a solution of the  $D = 4$  theory (1.1) provided the following relations between the parameters  $l, z$  and the coupling constants are obeyed

$$\beta_2 = -3\beta_1 = \frac{3l^2}{2z(z-4)}, \quad (4.2)$$

$$\Lambda = -\frac{z^2 + 2z + 3}{2l^2}. \quad (4.3)$$

That is, Lifshitz metric with arbitrary dynamical exponent  $z$  is a solution of  $D = 4$  square-curvature gravity with the precise relation between  $\beta_1$  and  $\beta_2$  that holds in Critical Gravity, namely (1.2). Nevertheless, when in addition to (1.2) one imposes the restriction (1.3) then one only finds  $z = 1$ , so that, at the critical point the Lifshitz solutions flow to the  $\text{AdS}_D$  spacetime. The same happens with the Schrödinger invariant backgrounds (2.16), which flow to the isotropic value  $z = 1$  when the critical point is approached. This phenomenon was also observed in three dimensions [30].

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